On the Statistics of Ultrasonically Produced Water Droplets

Dragan Šarkovic and Vukota Babovic

Physics Department, Faculty of Science, P. O. Box 60, 34000 Kragujevac, SCG

Reprint requests to Dr. V. B.; E-mail: bvukot@EUnet.yu

Z. Naturforsch. **60a,** 489 – 493 (2005); received June 3, 2003

We have developed a robust and efficient ultrasound liquid sprayer working at forty kilohertz. In the course of realizing the project of finding new ways in tobacco aromatization processes we met the problem of (approximate) analytical expressions of probability density functions for droplet sizes in various aerosols. This paper reports at what results we have arrived in that realm. Corresponding graphics of propability functions are presented.

Key words: Ultrasound Atomizer; Water Aerosol; Probability Function.

1. Introduction

In a recent paper [1] we have reported on the construction and functioning of an ultrasonically driven water atomizer. Its central part is the piezoceramics-based transducer resonating at the frequency $f =$ 40 kHz as the output organ of an electrical oscillator realized by means of two S 2530 H (silicon epitaxial planar power) transistors connected according to a standard push-pull scheme. We have proved that the apparatus can work as a robust and continuous producer of fine aerosol with a remarkable efficiency. In a number of conference papers [2] we have presented several aspects of various physical characterizations of water droplets constituting aerosol clouds. In another paper [3] we have summarized our experimental results on the most frequent droplet diameters; the measurements were based on microscopic observations followed by a CCD camera digitalizing procedure and PC image processing. We have also undertaken other independent measurements which include laser light diffraction on mists of aerosols. Our histograms, which present the percentage of droplets in a given interval of droplet sizes, follow the ones revealed by W. D. Drews ([4], for a frequency nearly three times ours). This is not a surprise – spraying of liquids is ‘a robust phenomenon’ observed in various fluids with over a two decade range in frequency [5].

The envelope of the distribution shows a maximum centered at D (the most probable diameter). The hypothesis which states that droplets are created from the surface of water via capillary waves (having wavelength λ_k at frequency f) is now the well-established theory. According to this idea, one could expect a dependence of the form $D = C^{-1}\lambda_\mathrm{k}(f)$, when the estimation of C could only experimentally be done. Originally, Pohlmann and Stamm [6] reasonably argued that $C = 4$, and later W. D. Drews confirmed that this is an acceptable coefficient of proportionality. Some more sophisticated approaches may require the constant of proportionality to be in the interval $3 < C < 4$, but this question is not crucial for our arguments in this paper and will be avoided here.

Our leading idea in the whole project was to develop a new, more prosperous method of aromatization in tobacco industry. In our first investigations we have tried to secure the reproducibility of the process; it was desirable to find an (at least approximate) analytical expression for the probability density function which characterizes the stochastic spraying process. In what follows we will expose our findings in that direction.

2. Probability Density Function

Our numerous droplet sizing measurements in the kilohertz region led us to the conviction that good fittings of experimentally deduced graphs could be achieved with a function

$$f(d) = ad^n \mathrm{e}^{-bd}. \tag{1}$$

Here, the three parameters a, b, and n could be so adjusted that $f(d)$ firmly follows the measured number of droplets in a narrow interval about the given value of the diameter d. As a matter of fact, we found that the choice $n = 3$ matches very well the original Drews’

0932–0784 / 05 / 0700–0489 $ 06.00

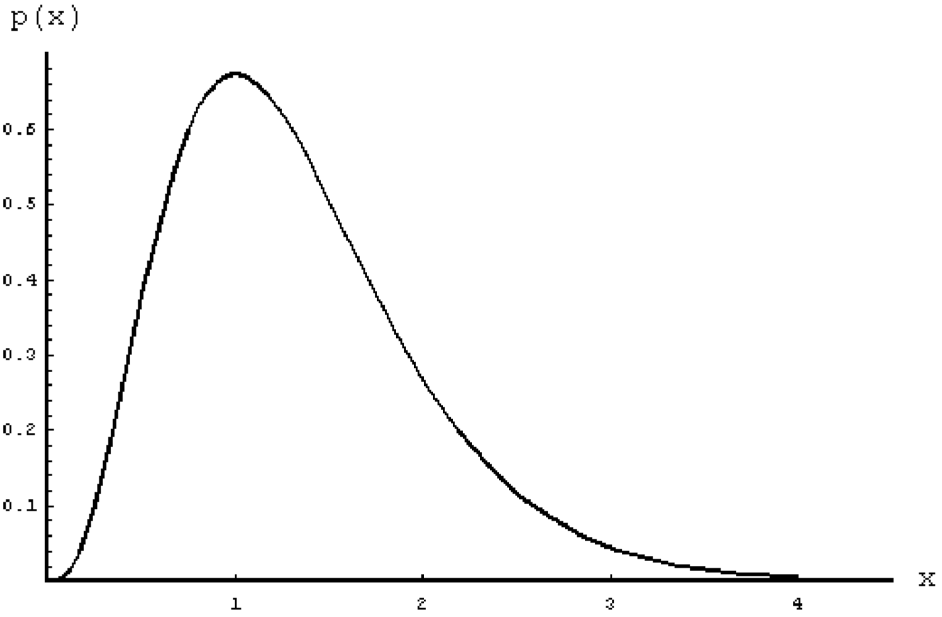

Fig. 1. Shape of normalized probability density function.

histograms, and ours too. By putting the normalization condition on the density function, i. e. demanding

$$\int_0^\infty f(d)\mathrm{d}d = 1, \tag{2}$$

we can straightforwardly obtain a connection between a and b of the type

$$a = \frac{b^4}{6}. \tag{3}$$

The mean diameter $\bar{d}$ may now be defined in the usual way via the density function

$$\bar{d} = \int_0^\infty d\, f(d)\mathrm{d}d. \tag{4}$$

Once again the integral can be evaluated in a closed form, and we get another condition of the form

$$\bar{d} = \frac{24a}{b^5}. \tag{5}$$

If one prefers to use the most probable value $d_\mathrm{m} \equiv D$ in the distribution, instead of the mean $\bar{d}$, the annulment of the first derivative gives $bD = 3$, from which follows the simple relation

$$D = \frac{3}{4}\bar{d}. \tag{6}$$

At least in principle, D and $\bar{d}$ are the measurable quantities. One has to decide which of the two is to be chosen as an experimentally verified parameter. The possibilities are: 1.) $a = 128/(3\bar{d}^4)$ and $b = 4/\bar{d}$ or 2.) $a = 27/(2D^4)$ and $b = 3/D$. Having (1) in mind we have proved that the probability density has the form

$$f(d) = \frac{128}{3\bar{d}^4} d^3 \exp\left(-4\frac{d}{\bar{d}}\right). \tag{7}$$

The expression derived via the parameter D reads

$$f(d) = \frac{27}{2D^4} d^3 \exp\left(-3\frac{d}{D}\right). \tag{8}$$

With $x = d/D$ we obtain the dimensionless probability density function

$$p(x) = Df(Dx) = \frac{27}{2}x^3\mathrm{e}^{-3x}, \tag{9}$$

which satisfies the physical condition $p(x)\mathrm{d}x = f(d)\mathrm{d}d$. Figure 1 shows the shape of this function.

The maximum height of this curve, located at $x = 1$, i. e. at $d = D$, amounts to $27/(2\mathrm{e}^3) \approx 0.672$. Half of this height is reached at $x_1 = 0.465$ and $x_2 = 1.842$, so $\Delta x = x_2 - x_1 = 1.377$. The corresponding interval of diameters amounts $\Delta d = d_2 - d_1 = 1.377D$ or

$$\Delta d \approx \bar{d}, \tag{10}$$

in view of the fact that $4D = 3\bar{d}$.

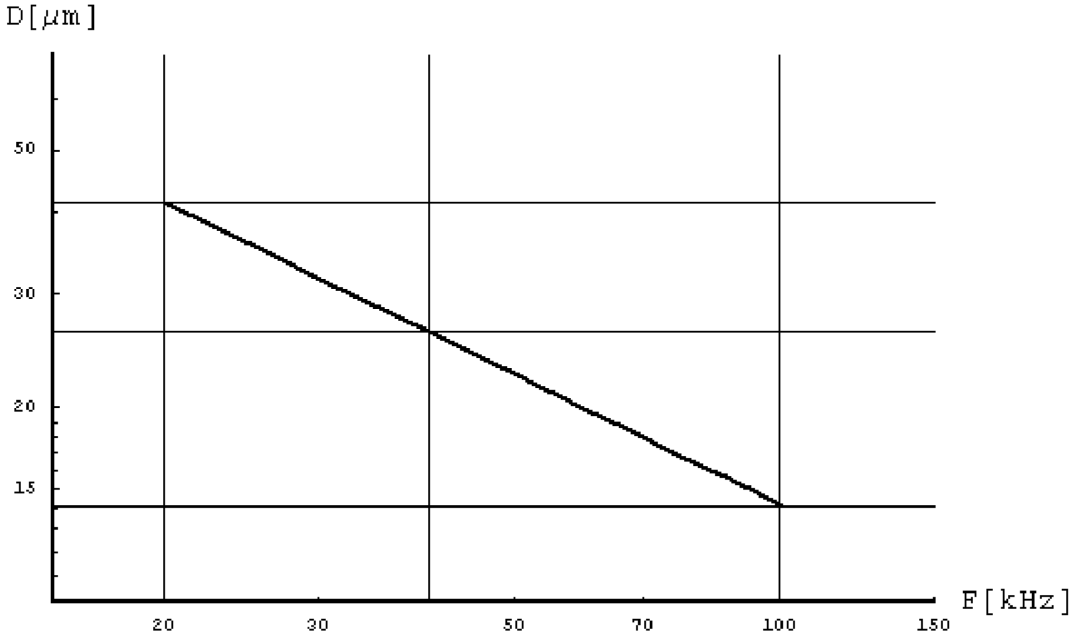

Fig. 2. Most probable water droplet size as a function of frequency; a log-log plot.

3. Frequency-dependent Probability Density Function

We shall now try to formulate a probability density function which explicitly depends on the driving frequency f. As we have already stressed, according to the idea that a droplet originates when an amount of liquid tears away from some crest in the (large amplitude) surface capillary wave field, one could expect in this stochastic process a dependence of the form $D = C^{-1}\lambda_k(f)$. The wavelength follows from the well-known dispersion equation of these waves:

$$\lambda_k(f) = 2\sqrt[3]{\pi}\sqrt[3]{\frac{\gamma}{\rho f^2}}. \tag{11}$$

Here γ stands for the coefficient of surface tension of a given liquid and ρ for its mass density.

We will accept Drews' recommendation $C = 4$ which gives

$$D = d_0 F^{-2/3}, \tag{12}$$

with the abbreviations $f_0 = 1$ kHz and $d_0 = 0.5\pi^{1/3}[\gamma/(\rho f_0^2)]^{1/3}$, wherein F is the frequency expressed in kilohertzes. For water, the above formulae could additionally be simplified by putting $\gamma = 0.073$ N/m, as well as $\rho = 1000$ kg/m^3. In this way we obtain the water normalizing parameter $d_0 = 300\ \mu$m. Under these circumstances, one may use the following simple and practical formula for water as working fluid

$$D \simeq 300F^{-2/3}. \tag{13}$$

Here, the mean diameter is expressed in microns. This result is graphically presented in Fig. 2, using a log-log scaling.

Ultrasonically produced water aerosol clouds contain a dominant group of droplets with diameters centered around 25 μm if the driving frequency is 40 kHz.

Having in mind (12), the distribution (9) can be put into the form

$$p(t,F) = \frac{27}{2}t^3F^2\mathrm{e}^{-3tF^{2/3}}, \tag{14}$$

where $t = d/d_0$ is the new normalized variable.

4. Relation to Generalized Incomplete Gamma Function

We shall now demonstrate that the probability density function we introduced in (9) can be related with the generalized incomplete Gamma function usually

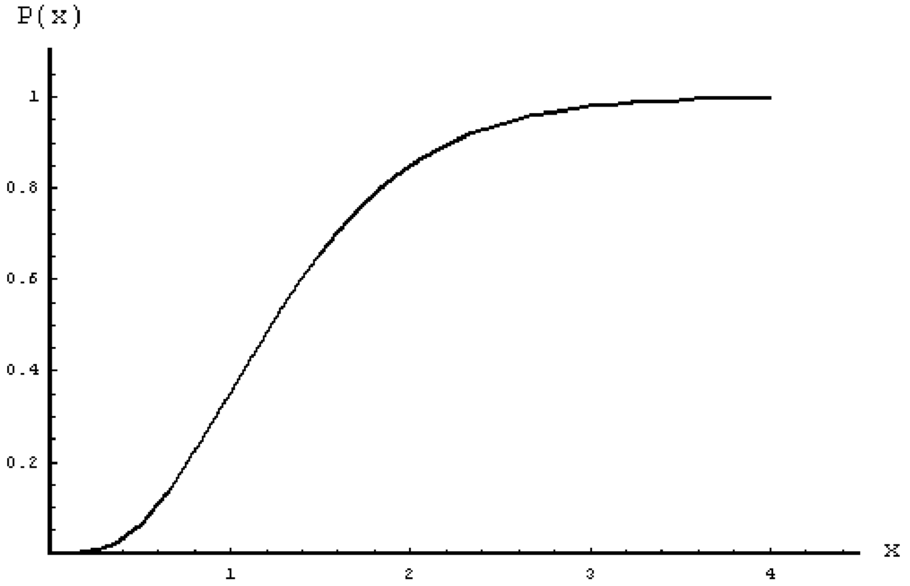

Fig. 3. Normalized cumulative distribution function.

defined as [7]

$$\Gamma(a,z_0,z_1) = \int_{z_0}^{z_1} t^{a-1}\mathrm{e}^{-t}\mathrm{d}t. \tag{15}$$

Really, let us attach to the set of parameters values $a = 4$, $z_0 = 0$ and $z_1 = 3x$. Then the function

$$p(x) = A\frac{d\Gamma(4,0,3x)}{\mathrm{d}x} \tag{16}$$

exactly matches the expression (9) if we put $A = 0.16$, i. e. the first derivative $\Gamma'(4,0,3x)$ is proportional to our probability density function. Taking into account that the probability density function itself is the first derivative of a cumulative distribution function, we conclude that the latter is defined as

$$P(x) = A\Gamma(4,0,3x). \tag{17}$$

Figure 3 shows this function in the relevant domain; $P(x)$ monotonically grows, practically reaching its saturation level at $x \sim 5$, i.e. for a droplet diameter not greater than several most probable diameters.

Nearly 80% of all droplets in a cloud have diameters not greater than $2D$; thus, for aerosols produced with a 40 kHz ultrasonic atomizer this theoretical model predicts $d_{80\%} < 50\ \mu$m.

5. Comments and Conclusion

Our theoretical model of the droplet size distribution in the core of an aerosol produced by an ultrasonic atomizer working in a frequency region up to 200 kHz leads to (16) for probability density function and (17) for cumulative distribution function. Primarily, this formulas are designed to be convenient for water aerosols under room conditions. Nevertheless, we believe the expressions are valuable for other liquids too, unless their viscosity coefficientes could not be meant as sufficiently small. The physical properties of a liquid (density and surface tension) enter the probability function via the theoretical-experimental binding of the type (11). Corresponding functions, here not explicitly written down, can be straightforwardly obtained.

One important effect, which could seriously affect the process of droplet emission from capillary wave peaks, is the amplitude of vertical liquid oscillations. Our attention was concentrated on relatively small amplitudes not considerably exceeding a starting point for emission. As Drews' reports, these are of several microns (more precisely, he mentioned in [4] 3 µm), and in good agreement with our observations. We thus restrict our theoretical results to a near-to-threshold domain of amplitudes. Our experiments have been usually performed keeping the efficiency of spraying under a modest level of 5 cm^3/min.

We believe the results here presented may be useful for people who are doing experiments, under the restrictions stressed, as an auxiliary tool when planning experiments, performing measurements and evaluating effects of spraying. What is more convenient, the mathematics needed in treating the distribution functions well-defined and already regularly included in contemporary software packages, what could facilitate manipulations in performing a given project.

[1] D. Šarkovic and V. Babovic, J. Sci. **24**, 41 (2002).
[2] D. Šarkovic and V. Babovic, 12th Scientific Conference on Industrial Systems, 271 (2002).
[3] D. Šarkovic and V. Babovic, Nis Facta Universitatis **2**, 197 (2002).
[4] W. D. Drews, Elektronik – München **10**, 83 (1979).
[5] C. L. Goodridge, W. Tao Shi, H. G. E. Hentschel, and D. P. Lathrop, Phys. Rev. E **56**, 472 (1997).
[6] R. Pohlmann and K. Stamm, Untersuchungen zum Mechanismus der Ultraschallvernebelung an Flüssigkeitsoberflächen in Hinblick auf technische Anwendungen, Westdeutscher Verlag Köln und Opladen 1965.
[7] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, Dover Publications, Inc., New York 1972.